

A Bond Market Model Based on Discrete Time/State Space Approximation of the Vasicek Model

Nedim Gavranović

American University in Bosnia and Herzegovina, Sarajevo, Bosnia and Herzegovina
nedim@bih.net.ba

Abstract

In this paper we present the mathematical model for the interest rate (either real or nominal) as an autoregressive discrete time and discrete state space process. Having defined an interest rate model with discrete time/state spaces, we derive zero-coupon prices for bonds with any duration and any initial value of the interest rate. The process is an approximation of Vasicek continuous time/state space autoregressive process presented in Vasicek (1977). We choose Vasicek model for interest rate for developing bond prices as the one which is used in the analysis of optimal asset allocation problems by many authors. It is a type of one factor short rate model where interest rate movements are driven by one source of market risk. Our model can be used in many applications when modeling an interest rate or bond prices mathematically. It is particularly suitable for making simulations on the computer. The shortcoming of Vasicek model is the positive probability of the negative value of interest rate. Due to mean reverting characteristic of the interest rate, even for the negative value of interest rate, there will be a certain demand for both traditional and index-linked bonds. It is possible to derive the bond market model using the interest rate which does not allow the negative values of the interest rate, for example Cox–Ingersoll–Ross model (Cox et al (1985)). Although CIR model may be deemed as a more appropriate, it would be also computationally more demanding. In our model we assume that the discrete time interval is one year. We will show the technique to transform the continuous time Vasicek process into a discrete time one. As the Vasicek process is transformed into discrete time process, it is still a continuous state space process. We use the technique from Tauchen and Hussey (1991) and as a result get a process with discrete time/state spaces. Once we obtain a discrete time/state process for interest rate we can model bond prices as the expected present value of future incomes from the bond. We model a zero coupon bond. Thus, the bond price is expected present value of one money unit that will be due in n years, where n years is the bond duration. Following the Vasicek approach, we can also introduce a market price of risk. As a final result we get the model for the zero-coupon bond prices for the whole bond market (different durations) and for different states of economy (different known values of the interest rate).

Keywords: interest rate; Vasicek model, AR(1) process; approximation; computer modeling; discrete time/state spaces, bond market model

Introduction

In different financial models, one needs to decide if the assumption of constant inflation or constant interest rate is an acceptable approximation. Namely, under this assumption the model does not recognize the risk of inflation or interest rate risk. Adding these risks in the model give us the new insight into the importance of these risks.

The usual assumption for the interest rate (or inflation) in the model is one of the following: constant, identically independently distributed (iid) random variables for each time period, discrete time stochastic process and continuous time stochastic process.

Often, continuous time models better represent the real world. The advantage of a discrete time model over continuous time one is the possibility to solve the problem on computers, and sometimes to obtain the results numerically while the analytical solution is not available with a current mathematical knowledge. In recent years we have witnessed the fast development of computer hardware and software, and of parallel computing. So, when we develop a discrete time model there are very powerful tools for obtaining a numerical solution. Even more, if we want to improve the model, for example to add certain constraints or to add annuities or one or more variables, the improved version of the model still can be solvable. A shortcoming of the numerical solution on the computer is that we usually get one numerical solution for one choice of the values for each parameter. In order to get an idea about the solution for different values of the parameters, we need to get a number of solutions and to compare them numerically.

We model the interest rate as an autoregressive discrete time and discrete state space process. The process is an approximation of Vasicek continuous time/state space autoregressive process presented in Vasicek (1977). As the Vasicek model provides bond prices for an implied bond market, we can compare bond prices on the bond market obtained in our model with the one obtained from Vasicek model.

Wilkie (1986, 1995) develops a discrete time and state spaces stochastic inflation model similar to our model presented here. Our approach is to start from Vasicek model and develop formulae directly from Vasicek model. For example, our approach can be applied to making discrete time and state spaces approximation of the bond market developed by Boulier et al (2001), and similar reasoning could be applied to the work of Deelstra et al (2000).

In our model we assume that the discrete time interval is one year. We assume that interest rate can take finite number of values in a reasonable range. Firstly, the Vasicek process is transformed into discrete time and a continuous state space process. Then, we use the technique from Tauchen and Hussey (1991) and as a result we get a process with discrete time/state space.

In Section 2, we present assumptions and the main parts of the Vasicek model. In Section 3, we start from the formulae provided in Vasicek model and derive formulae for discrete model of interest rate. Once we obtain a discrete time/state process for real interest rate we can model bond prices as the expected present value of future income. As we assume a zero coupon bond, it means that the bond price is expected present value of one money unit that will be due in n years, where n years is the bond duration. Following the Vasicek approach, we can also introduce a market price of risk. In Section 4, as a final result we get the approximation of the bond market. The model for the bond market is based on the discrete time/state space interest rate and can be used for computer simulation of the bond market that is consistent with the interest rate.

The Main Formulae of the Vasicek Model

The Vasicek model is used for modeling interest rate where time and state spaces are continuous. It is a continuous time AR (1) process given by

$$d\tilde{r}_t = (a - b\hat{r}_t)dt - \sigma_{\tilde{r}}d\tilde{W}_{\tilde{r}}(t) \quad (1)$$

where \hat{r}_0 is the initial value of the interest rate, a , b and $\sigma_{\tilde{r}}$ are non-negative constants and $\tilde{W}_{\tilde{r}}(t)$ is Brownian motion. We use notation \hat{r}_t for interest rate from Vasicek model in order to avoid the confusion with interest rate afterwards in this paper. As throughout the whole paper, the sign \square above variable denotes it is a random variable.

We know that \tilde{r}_t is a normally distributed random variable and that the conditional expectation and variance of the process given current level \hat{r}_0 are

$$E[\tilde{r}_T] = \frac{a}{b} + \left(\hat{r}_0 - \frac{a}{b}\right)e^{-bT} \quad (2)$$

$$Var[\tilde{r}_T] = \frac{\sigma_{\tilde{r}}^2}{2b}(1 - e^{-2bT}) \quad (3)$$

for $T \geq 0$.

The stochastic differential equation of the bond investments is given by

$$\frac{d\tilde{B}(T-t, \hat{r}_t)}{B(T-t, \hat{r}_t)} = (\hat{r}_t + \sigma_B(T-t, \hat{r}_t)\lambda_{\tilde{r}})dt + \sigma_B(T-t, \hat{r}_t)d\tilde{W}_{\tilde{r}}(t) \quad (4)$$

where t is the time such that $0 \leq t \leq T$, T is bond duration, $B(T, T) = 1$, and

$$\lambda_{\tilde{r}} = \frac{\mu_B(T-t, \hat{r}_t) - \hat{r}_t}{\sigma_B(T-t, \hat{r}_t)}.$$

$\lambda_{\tilde{r}}$ is referred to as bond's market price of risk and is constant. The function $\sigma_B(T-t, \hat{r}_t)$ is given by

$$\sigma_B(T-t, \hat{r}_t) = \frac{1 - e^{-b(T-t)}}{b} \sigma_{\tilde{r}} \quad (5)$$

for $T-t \geq 0$.

If we work with zero-coupon bonds and assume that we are interested in the value at time $t=0$ of the bonds maturing at time T and assuming current value of the interest rate is \hat{r}_0 , then the price of the zero-coupon bond is given by

$$B(T, \hat{r}_0) = \text{Exp} \left[-\frac{\sigma_{\hat{r}} \lambda_{\hat{r}}}{b} \left(\frac{1-e^{-bT}}{b} - T \right) \right] \cdot \text{Exp} \left[\left(\frac{1-e^{-bT}}{b} - T \right) \left(\frac{a}{b} - \frac{1}{2} \left(\frac{\sigma_{\hat{r}}}{a} \right)^2 \right) - \frac{1-e^{-bT}}{b} \hat{r}_0 - \frac{\sigma_{\hat{r}}^2}{4a^3} (1-e^{-bT})^2 \right] \quad (6)$$

Discrete Time/State Space Approximation of the Vasicek Model

In order to approximate Vasicek model in discrete time and continuous state space we observe the process

$$\Delta \tilde{R}_t = (a_d - b_d R_t) \Delta t - \sigma_{dr} \tilde{\varepsilon}_R(t) \quad (7)$$

where $\tilde{\varepsilon}_R(t) \square N(0,1)$ are independent random variables with normal distribution, for $t \in \square$. In order to have similar results from the continuous time and discrete time process we fit the parameters a_d , b_d and σ_{dr} into the Vasicek model (1).

Let us derive formula for R_T using equation (7). We have

$$\begin{aligned} \tilde{R}_1 - R_0 &= a_d - b_d R_0 - \sigma_{dr} \tilde{\varepsilon}_R(1) \text{ and} \\ \tilde{R}_1 &= a_d + (1-b_d)R_0 - \sigma_{dr} \tilde{\varepsilon}_R(1). \end{aligned}$$

Then

$$\begin{aligned} \tilde{R}_2 &= a_d + (1-b_d) \tilde{R}_1 - \sigma_{dr} \tilde{\varepsilon}_R(2) \\ &= a_d + (1-b_d) (a_d + (1-b_d)R_0 - \sigma_{dr} \tilde{\varepsilon}_R(1)) - \sigma_{dr} \tilde{\varepsilon}_R(2) \\ &= a_d \sum_{k=0}^{2-1} (1-b_d)^k + (1-b_d)^2 R_0 - \sigma_{dr} \sum_{k=1}^2 (1-b_d)^{2-k} \tilde{\varepsilon}_R(k) \end{aligned}$$

Continuing the similar reasoning gives us the relation

$$\tilde{R}_T = a_d \sum_{k=0}^{T-1} (1-b_d)^k + (1-b_d)^T R_0 - \sigma_{dr} \sum_{k=1}^T (1-b_d)^{T-k} \tilde{\varepsilon}_R(k), \text{ for } \forall T \in \square \quad (8)$$

Knowing that the sum of normally distributed random variables is again normally distributed random variable we have that

$$\sigma_{dr} \sum_{k=1}^T (1-b_d)^{T-k} \tilde{\varepsilon}_R(k) \square N \left(0, \sigma_{dr}^2 \sum_{k=1}^T (1-b_d)^{2(T-k)} \right), \text{ or}$$

Now, we can easily derive

$$E[\tilde{R}_T] = \frac{a_d}{b_d} + \left(R_0 - \frac{a_d}{b_d} \right) (1-b_d)^T \quad (9)$$

and

$$Var[\tilde{R}_T] = \sigma_{dR}^2 \frac{1 - (1 - b_d)^{2T}}{b_d(2 - b_d)} \quad (10)$$

Let us determine the coefficients a_d , b_d and σ_{dR} such that equations (2) and (9), and (3) and (10) respectively, gives the same values. From the first two equations, by equating the expectations, we have that

$$b_d = 1 - e^{-b} \quad (11)$$

and

$$a_d = a \frac{1 - e^{-b}}{b} \quad (12)$$

Now, from the second pair of equations, by equating variances, we get

$$\sigma_{dR} = \sigma_{\tilde{r}} \sqrt{\frac{1 - e^{-2b}}{2b}} \quad (13)$$

The discrete time version of the Vasicek process given in (7) is now fully defined and the appropriate parameter values are given in (11)–(13). We have the discrete time and continuous state AR (1) process such that \tilde{R}_t is normally distributed and the conditional expectation and variation of this random variable is the same as the conditional expectation and variance for the Vasicek process given in(1). Thus, we have defined the discrete time and continuous state space approximation of the Vasicek process(1).

Tauchen and Hussey (1991) gives the technique for approximating discrete time and continuous state space AR (1) process with a discrete time and state spaces process. We apply this technique to the process(7).

In order to deploy the technique from Tauchen and Hussey (1991), we need to choose the density function $\omega(y)$, and the number N denoting the number of Quadrature points. Let the density function $\omega(y)$ be the density function of the random variable with the distribution

$$N\left(\frac{a_d}{b_d}, \sigma_{dR}\right). \quad (14)$$

This choice is based on the proposal in Tauchen and Hussey (1991), where the authors say that this choice works well in most examples.

Let us denote with \tilde{r}_t random variable which has discrete time and state spaces and which approximate random variable \tilde{R}_t . It is autoregressive process defined in the form

$$P[\tilde{r}_{t+1} = r_{t+1,k} | \tilde{r}_t = r_{t,j}] \quad (15)$$

for $1 \leq j, k \leq N$. The constants $r_{t,i}$ for $1 \leq i \leq N$ are the possible states of the interest rate to be defined below.

Let the number of Quadrature points be N . The bigger the number of points the better is approximation. However, the choice of $N = 15$ provides quite good behavior and we show the results with the choice of 15 Quadrature points in Appendix.

Based on this choice we choose abscissa points, i.e. the possible states of the interest rate are constants $r_{t,1}, r_{t,2}, \dots, r_{t,N}$, such that the probabilities derived using this technique satisfies the condition $P[\tilde{r}_{t+1} = r_{t+1,i} | \tilde{r}_t = r_{t,i}] < 0.02$ and $P[\tilde{r}_{t+1} = r_{t+1,i} | \tilde{r}_t = r_{t,N}] < 0.02$ for $1 \leq i \leq N$ and that the points are derived from Gauss Quadrature with these ending points. We derive the weights w_1, \dots, w_N , for these choice of abscissa points and the density function $\omega(y)$.

Let us also define the function $f(y | r_0)$ as the density function for the random variable with the distribution

$$N \left(\frac{a_d}{b_d} + \left(r_0 - \frac{a_d}{b_d} \right) (1 - b_d), \sigma_{dr} \right) \quad (16)$$

Having determined the abscissa points, the weighting function and the function $f(y | r_0)$, we can apply the Tauchen and Hussey (1991) technique as follows. Let

$$s(r_j) = \sum_{i=1}^N \frac{f(r_i | r_j)}{\omega(r_j)} w_j \quad (17)$$

and let

$$\pi_{jk}^N = \frac{f(r_k | r_j)}{s(r_j) \omega(r_k)} w_k \quad (18)$$

Then according to Tauchen and Hussey (1991), we have

$$\left\{ \pi_{jk}^N \right\}_{(j,k)=(1,1)}^{(N,N)} = \left\{ p_{jk} \right\}_{(j,k)=(1,1)}^{(N,N)} = P[\tilde{r}_{t+1} = r_{t+1;k} | \tilde{r}_t = r_{t;j}] \quad (19)$$

Numerical Derivation of the Bond prices

In Section 3, we defined the discrete time/state spaces autoregressive process which approximates the Vasicek model. Now, we derive the zero-coupon bond prices from this process and get the model for the bond market.

We first derive the price of the zero-coupon bond with no market price of risk. As usual, it is defined as expected present value of one unit payout after time T . Thus, we have

$$\bar{B}(T, r_0) = E \left[e^{-\tilde{r}_1} e^{-\tilde{r}_2} \dots e^{-\tilde{r}_T} \right] \quad (20)$$

where \tilde{r}_1 is a random variable denoting random interest during the first year, \tilde{r}_2 is a random variable denoting random interest during the second year knowing \tilde{r}_1 , and so on. In order to allow for the existence of the market price of risk, we use the idea from equation (6) and introduce the market price of risk by multiplying the bond price with no market price of risk (equation (20)) with the similar factor as in the continuous time Vasicek model. Let the constant λ_r represents the market price of risk in the Vasicek bond market model. Then we get the equation for the price of a zero-coupon bond as follows

$$B(T, r_0) = e^{-\frac{\sigma_r \lambda_r}{b} \left(\frac{1}{b} (1 - e^{-bT}) - T \right)} E \left[e^{-\tilde{r}_1} e^{-\tilde{r}_2} \dots e^{-\tilde{r}_T} \right] \quad (21)$$

Let us explain how we can calculate numerically the bond price in discrete time/state spaces. Following the main formula for the expected value we have that

$$\begin{aligned} B(1, r_0 = r_{0;j}) &= e^{-\frac{\sigma_r \lambda_r}{b} \left(\frac{1 - e^{-b}}{b} - 1 \right)} E \left[e^{-\tilde{r}_1} \right] \\ &= e^{-\frac{\sigma_r \lambda_r}{b} \left(\frac{1 - e^{-b}}{b} - 1 \right)} \sum_{k=1}^N e^{-\tilde{r}_{1;k}} p_{jk} \end{aligned}$$

For the bond of the duration two years we have

$$B(2, r_0 = r_{0;j}) = e^{-\frac{\sigma_r \lambda_r}{b} \left(\frac{1 - e^{-2b}}{b} - 2 \right)} E \left[e^{-\tilde{r}_1} e^{-\tilde{r}_2} \right] = e^{-\frac{\sigma_r \lambda_r}{b} \left(\frac{1 - e^{-2b}}{b} - 2 \right)} \sum_{k_1=1}^N \left(\sum_{k_2=1}^N e^{-\tilde{r}_{1;k_1}} e^{-\tilde{r}_{2;k_2}} p_{k_1 k_2} \right) p_{jk_1}$$

or

$$B(2, r_0 = r_{0;j}) = e^{-\frac{\sigma_r \lambda_r}{b} \left(\frac{1 - e^{-2b}}{b} - 2 \right)} \sum_{k_1=1}^N e^{-\tilde{r}_{1;k_1}} \left(\sum_{k_2=1}^N e^{-\tilde{r}_{2;k_2}} p_{k_1 k_2} \right) p_{jk_1}$$

The same pattern is applied for longer durations. However, we can see that the part of the second sum is the same as the sum for the bond with one year duration. Apart from the coefficient for the market price of risk the difference is in the indices only. Using this observation, one can firstly calculate the prices of bonds with the duration of 1 year and for all possible states for r_0 and then use these results to obtain the results for the bond with duration of two years. This feature is important when the calculation is applied on the computer. If we define

$$\bar{B}(1, r_1 = r_{1;k_1}) = \sum_{k_2=1}^N e^{-\tilde{r}_{2;k_2}} p_{k_1 k_2} \quad (22)$$

Then one can write

$$B(2, r_0 = r_{0;j}) = e^{-\frac{\sigma_r \lambda_r}{b} \left(\frac{1 - e^{-2b}}{b} - 2 \right)} \sum_{k_1=1}^N e^{-\tilde{r}_{1;k_1}} \bar{B}(1, r_1 = r_{1;k_1}) p_{jk_1}$$

Similarly, if we define

$$\bar{B}(2, r_1 = r_{1:k_1}) = \sum_{k_2=1}^N e^{-r_{2:k_2}} \bar{B}(1, r_2 = r_{2:k_2}) p_{k_1 k_2}$$

then

$$B(3, r_0 = r_{0:j}) = e^{-\frac{\sigma_r \lambda_r}{b} \left(\frac{1-e^{-3b}}{b} - 3 \right)} \sum_{k_1=1}^N e^{-\bar{r}_{k_1}} \bar{B}(2, r_1 = r_{1:k_1}) p_{jk_1}.$$

Following this pattern, we get an inductive formula for bond prices which significantly reduces computing time.

However, we calculate bond prices $B(T, r_0 = r_{0:j})$, for $0 \leq T \leq T_{\max}$ and $1 \leq j \leq N$ only once and then use the results in the simulations. So, it is important to calculate it in reasonable time only once.

Future Research

We can use the model and its solution for the investigation of the influence of random inflation or random interest rate in different models. We can also use the results for the models where we need bond prices consistent with a stochastic interest rate. The results are particularly useful for making stochastic simulations on the computer.

We model the interest rate using AR(1) Vasicek model. Another model for the interest rate can be used for developing the values of the interest rate in discrete time/statespace environment. The similar technique could be applied to other models as well.

Appendix

In Appendix, we firstly derive the formula for the exact value of bond prices in discrete time and continuous state space. Then, we compare bond prices derived from the Vasicek model (continuous time/state spaces) with bond prices derive, from the first approximation of the Vasicek model (discrete time and continuous state spaces) and from the second approximation of the Vasicek model (discrete time/state spaces). This Appendix is intended to give the idea of the changes in bond prices due to the approximation. We will not try to evaluate the quality of approximation by any criteria, just to give comparable bond prices values.

Equation (20) for the discrete time and continuous state spaces AR(1) process (7) can be solved exactly. Having solved equation (20), we multiply it by the factor

$$e^{-\frac{\sigma_r \lambda_r}{b} \left(\frac{1-e^{-Tb}}{b} - T \right)} \quad (23)$$

for $T \in \mathbb{Q}$ and get the exact bond prices in the first approximation of the Vasicek model, where we have discrete time and continuous state spaces. For $T=1$ equation (20) in discrete time and continuous state spaces can be written as

$$\bar{B}(1, r_0) = E \left[e^{-\bar{r}_1} \right] = \int_{-\infty}^{\infty} e^{-r_1} f(r_1 | r_0) dr_1 \quad (24)$$

Knowing that \tilde{r}_1 is normally distributed with mean and variance defined in (9) and (10) respectively, we have that

$$B(1, r_0) = \frac{1}{2\pi\sqrt{\text{Var}[\tilde{r}_1 | r_0]}} \int_{-\infty}^{\infty} e^{-\tilde{r}_1} e^{-\frac{(\tilde{r}_1 - E[\tilde{r}_1 | r_0])^2}{2\text{Var}[\tilde{r}_1 | r_0]}} d\tilde{r}_1 \quad (25)$$

and

$$\begin{aligned} \bar{B}(2, r_0) &= E\left[e^{-\tilde{r}_1} e^{-\tilde{r}_2} | r_0\right] = E\left[e^{-\tilde{r}_2} E\left[e^{-\tilde{r}_1} | r_0\right] | r_0\right] \\ &= E\left[e^{-\tilde{r}_2} \bar{B}(1, r_0) | r_0\right] \end{aligned}$$

As we know that \tilde{r}_2 is normal random variable, we can derive the solution of the last equation. Having the solution $\bar{B}(2, r_0)$ and multiplying it with factor defined in (23) for $T = 2$ we get the bond price with the duration of two years for any $r_0 \in \mathbb{R}$. Continuing this process, we can calculate any $\bar{B}(T, r_0)$, for $T \in \mathbb{N}$. Multiplying $\bar{B}(T, r_0)$ with factor defined in (23) we get bond prices for any duration and any $r_0 \in \mathbb{R}$.

There is a requirement to have certain relations between bond prices if we want to have a sound model. One way to check the soundness of the bond market model is to compare bond prices derived using the three models for the interest rate. We expect that, for the same duration and for the same initial value of the interest rate, bond prices have similar values. The second important thing we need to have in order to deem the bond prices model as a sound one is to have the same pattern when bond prices are compared in each model. It means that we expect decreasing bond prices as the value of the interest rate during the previous year increases.

In Table 1, we present the prices of zero-coupon bonds with the duration of five and ten years and different values of the interest rate during the previous year, for discrete time and state spaces, for discrete time and continuous state space, and for the Vasicek model.

Table 1: Calculated bond prices for the following values of the parameters: $a = 0.012$, $b = 0.6$ $\sigma = 0.02$ and $\lambda_r = 0,1528$, and $a_d = 0.009023$, $b_d = 0.451188$ and $\sigma_d = 0.015262$. Number of the interest rate states $N = 15$, the end points for the abscissa are -2.44% and 6.44% .

	Interest rate	Duration 5 year			Duration 10 year		
		Discrete time/state spaces, numerical solution	Discrete time/state continuous state spaces	Continuou s time/state spaces, Vasicek	Discrete time/state spaces, numerical solution	Discrete time/state continuous state spaces	Continuou s time/state spaces, Vasicek
1	-2.44%	92.96	93.78	95.55	82.35	83.20	84.84
2	-2.21%	92.79	93.53	95.21	82.19	82.97	84.52
3	-1.81%	92.49	93.10	94.60	81.91	82.57	83.96
4	-1.25%	92.04	92.50	93.77	81.50	82.01	83.19
5	-0.56%	91.44	91.77	92.75	80.96	81.33	82.24
6	0.22%	90.73	90.93	91.59	80.30	80.54	81.16
7	1.09%	89.92	90.02	90.34	79.56	79.70	79.99
8	2.00%	89.06	89.08	89.05	78.77	78.83	78.80
9	2.91%	88.21	88.15	87.79	77.99	77.97	77.62
10	3.78%	87.42	87.27	86.59	77.27	77.15	76.51
11	4.56%	86.73	86.48	85.50	76.64	76.41	75.50
12	5.25%	86.17	85.79	84.57	76.12	75.77	74.64
13	5.81%	85.74	85.24	83.83	75.73	75.26	73.95
14	6.21%	85.46	84.84	83.30	75.47	74.90	73.46
15	6.44%	85.30	84.62	83.00	75.33	74.69	73.18

We see that long term expected values $a/b=0.02$ as well as $a_d/b_d=0.02$, as we expected. When we compare bond prices with the same duration in each row we see similar values. For the two presented values of the bond duration, we can see the biggest range of bond prices is for the Vasicek model and the lowest is for discrete time and state spaces. However, observing the columns for the first and for the second approximation of the Vasicek model we can say that bond prices behave quite reasonably in terms of changes as function of the value of the interest rate during the previous year.

In Table 2 we present the values of the rates of return on 10 year rolling bonds during one year assuming the value of the interest rate during the previous year being -1.25% and 2.00% .

Table 2: Rates on 10 year rolling bonds during one year assuming the value of the interest rate during the previous year is -1.25% and 2.00% , and the value of interest the rate in the following year given in the first column.

	Interest Rate	$\frac{B(9, r_1)}{B(10, -1.25\%)} - 1$ in %			$\frac{B(9, r_1)}{B(10, 2.00\%)} - 1$ in %		
		Discrete time/state spaces, numerical solution	Discrete time/continuous state spaces	Continuous time/state spaces, Vasicek	Discrete time/state spaces, numerical solution	Discrete time/continuous state spaces	Continuous time/state spaces, Vasicek
1	-2.44%	3.55	3.96	4.51	7.14	8.15	10.33
2	-2.21%	3.36	3.67	4.11	6.94	7.85	9.91
3	-1.81%	3.01	3.17	3.42	6.57	7.33	9.18
4	-1.25%	2.49	2.47	2.47	6.04	6.61	8.18
5	-0.56%	1.81	1.62	1.30	5.33	5.72	6.95
6	0.22%	0.98	0.65	-0.02	4.48	4.71	5.54
7	1.09%	0.05	-0.41	-1.45	3.51	3.61	4.03
8	2.00%	-0.94	-1.49	-2.92	2.49	2.48	2.48
9	2.91%	-1.92	-2.57	-4.37	1.48	1.36	0.95
10	3.78%	-2.83	-3.59	-5.74	0.54	0.30	-0.49
11	4.56%	-3.62	-4.51	-6.97	-0.28	-0.66	-1.79
12	5.25%	-4.27	-5.31	-8.03	-0.95	-1.49	-2.91
13	5.81%	-4.76	-5.94	-8.88	-1.46	-2.15	-3.81
14	6.21%	-5.09	-6.40	-9.48	-1.80	-2.62	-4.44
15	6.44%	-5.27	-6.66	-9.83	-1.99	-2.89	-4.80

We suppose here that at the beginning of the year we know the value of the interest rate in the previous year and that 10 year zero coupon bond is priced according to that value. This known value of the interest rate is written in the header, and we present examples for the two value $r_0 = -1.25\%$ and $r_0 = 2.00\%$. Then we suppose that during the following year the value of the interest rate r_1 appears to be as in the first column. At the end of the year we have the price of the 9 year bond and calculate the rate of return on 10 year rolling bonds by $B(9, r_1)/B(10, r_0) - 1$. We can see in Table 2 that the rates of return on 10 year rolling bond investment have the highest range of values for the Vasicek model, the lower for the first approximation and the lowest for the second approximation. It means that in our examples, the variability of bond investment rates is lower compared to the Vasicek model. However, at the same time we can see a regular behavior of returns for both approximations. If σ takes lower values than 0.02, then we get the rates on ten years rolling bond investment using approximations that are more similar to the rates calculated from the Vasicek model.

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