Nonlinear Transverse Vibrations of a Slightly Curved Beam Carrying Multiple Concentrated Masses: Primary Resonance

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Abstract: In this study, nonlinear vibrations of curved Euler-Bernoulli beams carrying arbitrarily placed concentrated masses have been investigated. Sag-to-span ratio of the beam, which was assumed to have sinusoidal curvature function at the beginning, was taken as 1/10. Equations of motion were obtained by using Hamilton Principle. Cubic nonlinear terms aroused at the mathematical model because of the elongations occurred during the vibrations of the simple-simple supported beam. Method of multiple scales, a perturbation technique, was used for solving the equations of motion about analytically. Natural frequencies were obtained for different numbers, sizes and locations of the masses as control parameters. Analytical solutions were found for primary resonance case. Frequency-amplitude and frequency-response graphs were drawn using different control parameters for these resonance cases. Stability of the solutions was investigated in detail.

Keywords: curved beam, nonlinear vibrations, concentrated mass.

Introduction

Many engineering problems such as bridges, rails, automotive industries, work pieces and machine elements can be modeled as curved beams. Before proceeding to our investigation on these beams, some researches made on the beam vibrations, both linear and nonlinear, must be mentioned. Some of these studies are such that, Rehfield (1974) derived the equations of motion of a shallow arch with an arbitrary rise function and studied the free vibrations approximately. Singh and Ali (1975) studied a moderately thick clamped beam with a sinusoidal rise function by adding the effects of transverse shear and rotary inertia. Nayfeh *et al*. (1979) developed a new method, which is a combination of perturbation method and numerical method, to be used in the analysis of forced vibrations. Using two beam elements one has three degree-of-freedom and other four, Krishnan and Suresh (1998) studied static and free vibration of curved beams. Taking account into the effect of shear deformation and rotary inertia, they determined frequencies of these beams. For a general state of non-uniform initial stress, Chen and Shen (1998) derived the virtual work expressions of initially stressed curved beams. They investigated the influence of arc segment angles, elastic foundation, and initial stresses on natural frequencies. Oz *et al*.(1998) examined a simply supported slightly curved beam resting on an elastic foundation with cubic non-linearities. Considering free-undamped and forced-damped vibrations, he analyzed the effects of the elastic foundation, axial stretching and curvature on the vibrations of the beams. Tarnopolskaya, De Hoog and Fletcher (1999) examined the vibrational behavior of beams with arbitrarily varying curvature and cross-section in the lower region of the spectrum. For a particular type of beam curvature and cross-section, they examined whether or not the mode transition takes place. Lacarbonara *et al*. (2002) developed open-loop nonlinear control strategy, and applied it to a hinged-hinged shallow arch. They assumed the beam subjected to a longitudinal end-displacement with frequency twice the frequency of the second mode (principal parametric resonance). Tien *et al.* (1994) studied the dynamics of a shallow arch subjected to harmonic excitation. In the presence of both external and 1:1 internal resonance, he examined the bifurcation behavior of the shallow arch system. Lacarbonara, Yabuno and Okhuma (2003) investigated experimentally the principal parametric resonance of the second mode of a simply supported first-mode buckled beam. By considering axial loads slightly above the first buckling load, they examined the frequency-response curves for different excitation amplitudes and the space-time characteristics of the nonlinear resonant motions. Nayfeh *et al*. (1999) studied to construct the nonlinear normal modes of a fixed-fixed buckled beam about its first post-buckling mode. Abe (2006) studied the validity of nonlinear vibration analysis of continuous systems with quadratic and cubic nonlinearities. Lee, Poon and Ng (2006) studied to derive the equations of motion for a clamped–clamped curved beam subjected to transverse sinusoidal loads. Taking into account the effects of beam mid-plane stretching and damping Nayfeh and Pakdemirli (1994) investigated the nonlinear vibrations of a beam-mass-spring system. In their analysis frequency-response and force-response curves shows that the nonlinearity arouses due to stretching and location of nonlinear spring supporting the mass. Posiadala (1997) presented the solution of the free vibration problem of a Timoshenko beam with additional attached elements. By using the Lagrange multiplier formalism, he showed the influence of the various parameters on the frequencies of the combined system. Ozkaya *et al.* (1997) studied nonlinear vibrations of a beam-mass system under different boundary conditions. For different boundary conditions, locations and magnitude of the masses, he examined the effects of mid-plane stretching on the beam vibrations. Assuming simply supported end conditions, Ozkaya (2001) studied an Euler–Bernoulli beam carrying concentrated masses. He investigated the effects of mid-plane stretching on free-undamped and forced-damped vibrations of the beam in detail. Under assumption of simply supported end conditions Ozkaya (2002) studied nonlinear vibrations of an Euler–Bernoulli beam carrying concentrated masses. He investigated free-undamped and forced-damped vibrations of this beam–mass system for different locations, magnitudes and number of the masses. Adessi *et al.* (2005) studied the regime of high pre-stressed beams. Considering a lumped mass that is rigidly clamped to the beam at an arbitrary point along its span and assuming different boundary conditions(simply supported and hinged-hinged), they examined post-buckling configurations of the beam. The effect of the point concentrated mass on the large amplitude free vibrations of beam under symmetric configuration was investigated. Zhou and Ji (2006) studied free vibration characteristics of a non-uniform beam with arbitrarily distributed spring-mass. For the special cases of the proposed solution, they investigated the coupled vibrations of a beam and distributed spring-mass in detail. Hassanpour *et al.* (2007) investigated the vibrations of a beam with a concentrated mass within its interval length subjected to a quasi-static axial force. By choosing the location of the concentrated mass arbitrarily, they studied the transient and steady state behavior of the resonator in the time domain. Maiza *et al.* (2007) studied to describe the determination of the natural frequencies of a Bernoulli–Euler beam with general boundary conditions at the ends and carrying a finite number of masses at arbitrary positions, by considering their rotatory inertia. To present a general solution of the problem, they used translational and rotational springs at both ends as well as elastic restraints. Sochacki (2008) considered a simply supported beam loaded by both a longitudinal force and a concentrated mass in a chosen position along the beam length. He investigated the influence of additional mass and elasticity as well as an undamped harmonic oscillator on the position of the solutions on the stability chart. By considering, a continuous beam attached spring–mass systems and using directly differential equation of motion, Lin and Tsai (2007) obtained the natural frequencies and associated mode shapes of the vibrating system. They used FEA and thus made no other assumptions. Yesilce and Demirdag (2008) studied the multi-span uniform Timoshenko beam carrying multiple spring-mass systems with/without axial force effect. They described the determination of the natural frequencies and mode shapes of vibration as well as the effect of axial force. Finally, nonlinear transverse vibrations of a slightly curved Euler Bernoulli beam carrying a concentrated mass has been studied by E. Ozkaya *et al*. (2009)

In this study, nonlinear vibrations of curved beams carrying multiple concentrated masses were investigated. For the beam which is of Euler-Bernoulli type, it was assumed firstly that the beam had the form of sinusoidal rising function and was constricted from both ends by the immovable simply supports. The method of multiple scales (MMS), a perturbation method, was used in order to seek analytical solutions for the derived mathematical model. The primary resonance was investigated. Natural frequencies were calculated according to different control parameters such as number, magnitude and position of the masses. Amplitude and phase modulation equations were derived. Effects of the addition of nonlinear terms to the natural frequency were searched via frequency-amplitude and frequency-response graphs. Experiencing different control parameters, responses to the excitations were investigated. Having obtained solutions, the stable and unstable regions of the system were determined by using the stability analysis.

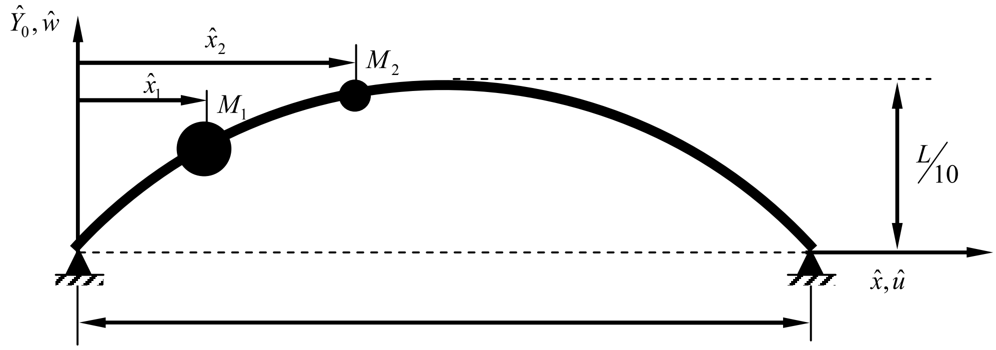


Figure 1. The curved beam carrying multiple concentrated masses.

Equations of motion

In Fig. 1, for the beam constricted at both ends with immovable supports, $\hat{w}_m$ and $\hat{u}_m$ denote transversal and longitudinal displacements, respectively. Assuming that ratio of the maximum amplitude of the beam to its projected length L is equal *1/10*, let us keep in mind the curvature function of the beam to be in the form of sinusoidal variation as given below:

$$Y_0(\hat{x}) = \frac{L}{10}.\sin\left(\pi.\frac{\hat{x}}{L}\right) \tag{1}$$

Let us assume that n number of concentrated masses is attached on the beam. The following equation and boundary conditions providing this equation can be written:

$$\rho.A.\ddot{\hat{w}}_{m+1} + E.I.\hat{w}_{m+1}'''' = \frac{E.A.}{L}\left[\sum_{r=0}^{n}\int_{\hat{x}_r}^{\hat{x}_{r+1}}\left\{\hat{Y}_0'.\hat{w}_{r+1}' + \frac{1}{2}\hat{w}_{r+1}'^2\right\}.d\hat{x}\right].\left(\hat{Y}_0'' + \hat{w}_{m+1}''\right), \qquad m = 0,\ 1 \ldots\ n. \tag{4.a}$$

$$\hat{w}_p\big|_{\hat{x}=\hat{x}_p} = \hat{w}_{p+1}\big|_{\hat{x}=\hat{x}_p},\ \hat{w}_p'\big|_{\hat{x}=\hat{x}_p} = \hat{w}_{p+1}'\big|_{\hat{x}=\hat{x}_p},\ \hat{w}_p''\big|_{\hat{x}=\hat{x}_p} = \hat{w}_{p+1}''\big|_{\hat{x}=\hat{x}_p},\ E.I.\left(\hat{w}_p''' - \hat{w}_{p+1}'''\right)\big|_{\hat{x}=\hat{x}_p} = M_p.\ddot{\hat{w}}_p\big|_{\hat{x}=\hat{x}_p}$$

$$\hat{w}_1\big|_{\hat{x}=\hat{x}_0} = \hat{w}_1''\big|_{\hat{x}=\hat{x}_0} = \hat{w}_{n+1}\big|_{\hat{x}=\hat{x}_{n+1}} = \hat{w}_{n+1}''\big|_{\hat{x}=\hat{x}_{n+1}} = 0, \qquad p = 1,\ 2 \ldots\ n. \tag{4.b}$$

where M is the concentrated mass attached on the beam, $\hat{x}$ is the distance from the immovable end at left-hand side, E is the Young's modulus, ρ is the density, A is the cross sectional area of the beam, I is the moment of inertia of the beam cross-section with respect to the neutral axis. $(\dot{\ })$ and $(\,'\,)$ denote differentiations with respect to the time t and spatial variable x, respectively.

Eq. (4.a) is the equation of motion for the system and consists of $n+1$ equations. Equations of the motion and the boundary conditions are dependent on the size of the system and the material used. These equations can be made independent from the dimensional parameters by making the following definitions:

$$w_p = \hat{w}_p/r,\quad Y_0 = \hat{Y}_0/r,\quad x = \hat{x}/L,\quad \eta_p = \hat{x}_p/L,\quad t = \sqrt{E.I/\rho.A.L^2}\,\hat{t},\quad \alpha_p = M_p/(\rho.A.L),\quad I = r^2.A \tag{5}$$

where r is the radius of gyration of the beam cross section, α is the ratio between the concentrated mass and the mass of the beam, η is the dimensionless displacement variable.

Adding dimensionless damping and forcing terms after non-dimensionalization, Eq. (4) can be rewritten as follows:

$$\ddot{w}_{m+1} + w_{m+1}^{iv} + 2.\vec{\mu}.\dot{w}_{m+1} = \left[\sum_{r=0}^{n}\int_{\eta_r}^{\eta_{r+1}}\left\{Y_0'.w_{r+1}' + \frac{1}{2}w_{r+1}'^2\right\}.dx\right].\left(Y_0'' + w_{m+1}''\right) + \vec{F}_{m+1}.\cos\Omega.t\,, \tag{6.a}$$

$$w_p\big|_{x=\eta_p} = w_{p+1}\big|_{x=\eta_p},\ w_p'\big|_{x=\eta_p} = w_{p+1}'\big|_{x=\eta_p},\ w_p''\big|_{x=\eta_p} = w_{p+1}''\big|_{x=\eta_p},\ \left(w_p''' - w_{p+1}'''\right)\big|_{x=\eta_p} = \alpha_p.\ddot{w}_p\big|_{x=\eta_p},$$

$$w_1\big|_{x=\eta_0} = w_1''\big|_{x=\eta_0} = w_{n+1}\big|_{x=\eta_{n+1}} = w_{n+1}''\big|_{x=\eta_{n+1}} = 0.\quad \eta_0 = 0,\ \eta_{n+1} = 1. \tag{6.b}$$

where μ is the dimensionless damping coefficient, F and Ω are the amplitude and frequency of the dimensionless external forcing term, respectively. In a similar way, the curvature function of the beam can be written in the following non-dimensional form:

$$Y_0(x) = sin(\pi.x) \tag{7}$$

Perturbation Analysis

In this section, approximate solutions to the system will be searched. Method of multiple scales (MMS), a perturbation technique, will be applied to the partial differential equations and corresponding boundary conditions directly. Eq. (6) is assumed to have a solution as a series expansion of the form below:

$$w_{m+1}(x,t;\varepsilon) = \sum_{j=1}^{3} \varepsilon^j . w_{(m+1)j}(x,T_0,T_1,T_2) + ... \tag{8}$$

where ε is a small bookkeeping parameter artificially inserted into the equations. Taking this parameter as 1 at the end, we obtain a weakly nonlinear system. In this expansion, $T_0 = t$ is the fast time scale, and $T_1 = \varepsilon t$ and $T_2 = \varepsilon^2 t$ are the slow time scales in MMS. Derivatives with respect to time are written as:

$$d/dt = D_0 + \varepsilon.D_1 + \varepsilon^2.D_2 + ..., \qquad d^2/dt^2 = D_0^2 + 2.\varepsilon.D_0.D_1 + \varepsilon^2.(D_1^2 + 2.D_0.D_2) + ... \quad D_n \equiv \partial/\partial T_n, \tag{9}$$

First order (ε^1) of the expansion in Eq. (9) corresponds to the linear problem of the system. Other orders constitutes nonlinear problem of the system. In order to counter the effects of the nonlinear terms, the forcing and damping terms are ordered as follows:

$$\vec{\mu} = \varepsilon^2.\mu, \qquad \vec{F}_{p+1} = \varepsilon^3.F_{p+1} \tag{10-11}$$

Let us assume that the curvature function is of order 1 (ε^0). In this case, substituting Eqs. (8-11) into Eq. (6) and separating each order of ε, one obtains the following equations:

order ε (j=1):

$$D_0^2.w_{(m+1)1} + w_{(m+1)1}^{iv} = \left\{ \sum_{r=0}^{n} \int_{\eta_r}^{\eta_{r+1}} Y_0'.w_{(r+1)1}' dx \right\}.Y_0'' \tag{12.a}$$

$$w_{p1}\big|_{x=\eta_p} = w_{(p+1)1}\big|_{x=\eta_p}, \quad w_{p1}'\big|_{x=\eta_p} = w_{(p+1)1}'\big|_{x=\eta_p}, \quad w_{p1}''\big|_{x=\eta_p} = w_{(p+1)1}''\big|_{x=\eta_p}$$

$$\left(w_{p1}''' - w_{(p+1)1}''' = \alpha_p.D_0^2.w_{p1} \right)\Big|_{x=\eta_p}, \quad w_{11}\big|_{x=\eta_0} = w_{11}''\big|_{x=\eta_0} = w_{(n+1)1}\big|_{x=\eta_{n+1}} = w_{(n+1)1}''\big|_{x=\eta_{n+1}} = 0, \tag{12.b}$$

order ε^2 (j=2):

$$\begin{aligned} D_0^2.w_{(m+1)2} + w_{(m+1)2}^{iv} = -2.D_0.D_1.w_{(m+1)1} + \left\{ \sum_{r=0}^{n} \int_{\eta_r}^{\eta_{r+1}} Y_0'.w_{(r+1)2}' dx \right\}.Y_0'' + \frac{1}{2}.\left\{ \sum_{r=0}^{n} \int_{\eta_r}^{\eta_{r+1}} w_{(r+1)1}'^2 dx \right\}.Y_0'' \\ + \left\{ \sum_{r=0}^{n} \int_{\eta_r}^{\eta_{r+1}} Y_0'.w_{(r+1)1}' dx \right\}.w_{(m+1)1}'' \end{aligned} \tag{13.a}$$

$$w_{p2}\big|_{x=\eta_p} = w_{(p+1)2}\big|_{x=\eta_p}, \quad w_{p2}'\big|_{x=\eta_p} = w_{(p+1)2}'\big|_{x=\eta_p}, \quad w_{p2}''\big|_{x=\eta_p} = w_{(p+1)2}''\big|_{x=\eta_p}$$

$$\left(w_{p2}''' - w_{(p+1)2}''' = \alpha_p.\left(D_0^2.w_{p2} + 2.D_0.D_1.w_{p1}\right)\right)\Big|_{x=\eta_p}, w_{12}\big|_{x=\eta_0} = w_{12}''\big|_{x=\eta_0} = w_{(n+1)2}\big|_{x=\eta_{n+1}} = w_{(n+1)2}''\big|_{x=\eta_{n+1}} = 0 \tag{13.b}$$

order ε^3 (j=3):

$$D_0^{\ 2}.w_{(m+1)3} + w_{(m+1)3}^{\ iv} = -2.\mu.D_0.w_{(m+1)1} - 2.D_0.D_1.w_{(m+1)2} - (D_1^{\ 2} + 2.D_0.D_2).w_{(m+1)1} + F_{m+1}.cos\Omega t$$

$$+\left\{\sum_{r=0}^{n}\int_{\eta_r}^{\eta_{r+1}} Y_0^{\ '}.w_{(r+1)3}^{\ '}\,dx\right\}.Y_0^{\ ''} + \left\{\sum_{r=0}^{n}\int_{\eta_r}^{\eta_{r+1}} w_{(r+1)1}^{\ '}.w_{(r+1)2}^{\ '}\,dx\right\}.Y_0^{\ ''} + \left\{\sum_{r=0}^{n}\int_{\eta_r}^{\eta_{r+1}} Y_0^{\ '}.w_{(r+1)2}^{\ '}\,dx\right\}.w_{(m+1)1}^{\ ''} \tag{14.a}$$

$$+\frac{1}{2}.\left\{\sum_{r=0}^{n}\int_{\eta_r}^{\eta_{r+1}} w_{(r+1)1}^{\ '2}.dx\right\}.w_{(m+1)1}^{\ ''} + \left\{\sum_{r=0}^{n}\int_{\eta_r}^{\eta_{r+1}} Y_0^{\ '}.w_{(r+1)1}^{\ '}\,dx\right\}.w_{(m+1)2}^{\ ''}$$

$$w_{p3}\big|_{x=\eta_p} = w_{(p+1)3}\big|_{x=\eta_p},\quad w_{p3}^{\ '}\big|_{x=\eta_p} = w_{(p+1)3}^{\ '}\big|_{x=\eta_p},\quad w_{p3}^{\ ''}\big|_{x=\eta_p} = w_{(p+1)3}^{\ ''}\big|_{x=\eta_p}$$

$$\left(w_{pj}^{\ '''} - w_{(p+1)j}^{\ '''}\right)\Big|_{x=\eta_p} = \alpha_p.\left(D_0^{\ 2}.w_{p3} + 2.D_0.D_1.w_{p2} + \left(D_1^{\ 2} + 2.D_0.D_2\right)w_{p1}\right)\Big|_{x=\eta_p},$$

$$w_{13}\big|_{x=\eta_0} = w_{13}^{\ ''}\big|_{x=\eta_0} = w_{(n+1)3}\big|_{x=\eta_{n+1}} = w_{(n+1)3}^{\ ''}\big|_{x=\eta_{n+1}} = 0, \tag{14.b}$$

Primary Resonance Case

Primary resonance occurs in case that the forcing frequency is close to one of the natural frequencies of the system. Thus, a sudden arise in the vibration amplitude happens. In order to solve linear problem in Eq. (12), we assume the solutions at order ε as of the following form:

$$w_{(m+1)1}(x,T_0,T_1,T_2) = \left[A(T_1,T_2).e^{i.\omega.T_0} + cc\right]Y_{m+1}(x) \tag{15}$$

where cc is the complex conjugate of the preceding terms, and ω is the natural frequency, Y_{m+1} are the functions describing the mode shapes. Inserting Eq. (15) into Eq. (12), following differential equations and boundary conditions can be obtained:

$$Y_{m+1}^{\ iv} - \omega^2.Y_{m+1} = \left\{\sum_{r=0}^{n}\int_{\eta_r}^{\eta_{r+1}} Y_0^{\ '}.Y_{r+1}^{\ '}\,dx\right\}.Y_0^{\ ''} \tag{16.a}$$

$$Y_p\big|_{x=\eta_p} = Y_{p+1}\big|_{x=\eta_p},\quad Y_p^{\ '}\big|_{x=\eta_p} = Y_{p+1}^{\ '}\big|_{x=\eta_p},\quad Y_p^{\ ''}\big|_{x=\eta_p} = Y_{p+1}^{\ ''}\big|_{x=\eta_p},\quad \left(Y_p^{\ '''} - Y_{p+1}^{\ '''} + \alpha_p.\omega^2.Y_p\right)\Big|_{x=\eta_p} = 0,$$

$$Y_1\big|_{x=\eta_0} = Y_1^{\ ''}\big|_{x=\eta_0} = Y_{n+1}\big|_{x=\eta_{n+1}} = Y_{n+1}^{\ ''}\big|_{x=\eta_{n+1}} = 0 \tag{16.b}$$

In order to obtain the solutions at order ε^2 of the perturbation series, it is required that a solvability condition such as $D_1A\ (T_1,\ T_2) = 0$ must be satisfied. Thus, the amplitude $A=A\ (T_2)$ does not depend on T_1. For obtaining the solution resulting from non-secular terms, Eq. (15) must be inserted into Eq. (13). In this case, equations at order ε^2 accept solutions of the form as below:

$$w_{(m+1)2}(x,T_2) = \left[A^2.e^{2.i.\omega.T_0} + cc\right]\phi_{(m+1)1}(x) + 2.A.\bar{A}.\phi_{(m+1)2}(x) \tag{17}$$

Inserting Eq. (17) into Eq. (13), differential equations and boundary conditions can be written as follows:

$$\phi_{(m+1)1}^{\ iv} - 4.\omega^2.\phi_{(m+1)1} = \left\{\sum_{r=0}^{n}\int_{\eta_r}^{\eta_{r+1}} Y_0^{\ '}.\phi_{(r+1)1}^{\ '}\,dx\right\}.Y_0^{\ ''} + \frac{1}{2}.\left\{\sum_{r=0}^{n}\int_{\eta_r}^{\eta_{r+1}} Y_{r+1}^{\ '2}\,dx\right\}.Y_0^{\ ''} + \left\{\sum_{r=0}^{n}\int_{\eta_r}^{\eta_{r+1}} Y_0^{\ '}.Y_{r+1}^{\ '}\,dx\right\}.Y_{m+1}^{\ ''} \tag{18.a}$$

$$\phi_{p1}\big|_{x=\eta_p} = \phi_{(p+1)1}\big|_{x=\eta_p},\quad \phi_{p1}^{\ '}\big|_{x=\eta_p} = \phi_{(p+1)1}^{\ '}\big|_{x=\eta_p},\quad \phi_{p1}^{\ ''}\big|_{x=\eta_p} = \phi_{(p+1)1}^{\ ''}\big|_{x=\eta_p},$$

$$\left(\phi_{p1}^{\ '''} - \phi_{(p+1)1}^{\ '''} + 4.\alpha_p.\omega^2.\phi_{p1}\right)\Big|_{x=\eta_p} = 0,\quad \phi_{11}\big|_{x=\eta_0} = \phi_{11}^{\ ''}\big|_{x=\eta_0} = \phi_{(n+1)1}\big|_{x=\eta_{n+1}} = \phi_{(n+1)1}^{\ ''}\big|_{x=\eta_{n+1}} = 0, \tag{18.b}$$

$$\phi_{(m+1)2}^{\ iv} = \left\{\sum_{r=0}^{n}\int_{\eta_r}^{\eta_{r+1}} Y_0^{\ '}.\phi_{(r+1)2}^{\ '}\,dx\right\}.Y_0^{\ ''} + \frac{1}{2}.\left\{\sum_{r=0}^{n}\int_{\eta_r}^{\eta_{r+1}} Y_{r+1}^{\ '2}\,dx\right\}.Y_0^{\ ''} + \left\{\sum_{r=0}^{n}\int_{\eta_r}^{\eta_{r+1}} Y_0^{\ '}.Y_{r+1}^{\ '}\,dx\right\}.Y_{m+1}^{\ ''} \tag{19.a}$$

$$\phi_{p2}\big|_{x=\eta_p}=\phi_{(p+1)2}\big|_{x=\eta_p},\quad \phi_{p2}{}'\big|_{x=\eta_p}=\phi_{(p+1)2}{}'\big|_{x=\eta_p},\quad \phi_{p2}{}''\big|_{x=\eta_p}=\phi_{(p+1)2}{}''\big|_{x=\eta_p},$$
$$\left(\phi_{p2}{}'''-\phi_{(p+1)2}{}'''\right)\Big|_{x=\eta_p}=0,\quad \phi_{12}\big|_{x=\eta_0}=\phi_{12}{}''\big|_{x=\eta_0}=\phi_{(n+1)2}\big|_{x=\eta_{n+1}}=\phi_{(n+1)2}{}''\big|_{x=\eta_{n+1}}=0 \tag{19.b}$$

At order ε^3 of the perturbation series, having substituted Eqs. (15-17) into Eq. (14), the resulting equation will accept the solution of the following form:

$$w_{(m+1)3}(x,T_0,T_2)=\varphi_{m+1}(x,T_2).e^{i.\omega.T_0}+W_{m+1}(x,T_2)+cc \tag{20}$$

where $W_{m+1}(x,T_2)$ corresponds to the solution for the non-secular terms, and cc to the complex conjugate of the preceding terms.
Excitation frequency is taken close to any natural frequency of the system as below:

$$\Omega=\omega+\varepsilon^2.\sigma \tag{21}$$

where σ is the detuning parameter denoting closeness of the forcing frequency to the natural frequency. Under this assumption, inserting Eq. (20) into Eq. (14) and eliminating the secular terms, the following differential equations and boundary conditions can be obtained:

$$\begin{aligned}
&\varphi_{m+1}{}^{iv}-\omega^2.\varphi_{m+1}-\left\{\sum_{r=0}^{n}\int_{\eta_r}^{\eta_{r+1}}Y_0{}'.\varphi_{r+1}{}'\,dx\right\}.Y_0{}''=-2.i.\omega.\left(\dot{A}+\mu.A\right)Y_{m+1}+\frac{1}{2}F_{m+1}.e^{i.\sigma.T_2}+\Bigg[\frac{3}{2}.\left\{\sum_{r=0}^{n}\int_{\eta_r}^{\eta_{r+1}}Y_{r+1}{}'^{2}.dx\right\}.Y_{m+1}{}''\\
&+\left\{\sum_{r=0}^{n}\int_{\eta_r}^{\eta_{r+1}}Y_{r+1}{}'.\phi_{(r+1)1}{}'\,dx\right\}.Y_0{}''+2.\left\{\sum_{r=0}^{n}\int_{\eta_r}^{\eta_{r+1}}Y_{r+1}{}'.\phi_{(r+1)2}{}'\,dx\right\}.Y_0{}''+\left\{\sum_{r=0}^{n}\int_{\eta_r}^{\eta_{r+1}}Y_0{}'.\phi_{(r+1)1}{}'\,dx\right\}.Y_{m+1}{}''\\
&+2.\left\{\sum_{r=0}^{n}\int_{\eta_r}^{\eta_{r+1}}Y_0{}'.\phi_{(r+1)2}{}'\,dx\right\}.Y_{m+1}{}''+\left\{\sum_{r=0}^{n}\int_{\eta_r}^{\eta_{r+1}}Y_0{}'.Y_{r+1}{}'\,dx\right\}.\phi_{(m+1)1}{}''+2.\left\{\sum_{r=0}^{n}\int_{\eta_r}^{\eta_{r+1}}Y_0{}'.Y_{r+1}{}'\,dx\right\}.\phi_{(m+1)2}{}''\Bigg].A^2.\bar{A}
\end{aligned} \tag{22.a}$$

$$\varphi_p\big|_{x=\eta_p}=\varphi_{p+1}\big|_{x=\eta_p},\quad \varphi_p{}'\big|_{x=\eta_p}=\varphi_{p+1}{}'\big|_{x=\eta_p},\quad \varphi_p{}''\big|_{x=\eta_p}=\varphi_{p+1}{}''\big|_{x=\eta_p}$$
$$\left(\varphi_p{}'''-\varphi_{p+1}{}'''+\alpha_p.\omega^2.\varphi_p\right)\Big|_{x=\eta_p}=2.i.\alpha_p.\omega.Y_p\big|_{x=\eta_p},\quad \varphi_1\big|_{x=\eta_0}=\varphi_1{}''\big|_{x=\eta_0}=\varphi_{n+1}\big|_{x=\eta_{n+1}}=\varphi_{n+1}{}''\big|_{x=\eta_{n+1}}=0 \tag{22.b}$$

The solvability condition for Eq. (22) can be written as follows:

$$2.i.\omega.\left[k.\dot{A}+\mu.A\right]+A^2.\bar{A}.\Gamma=\frac{1}{2}.f.e^{i.\sigma.T_2} \tag{23}$$

where normalization process and coefficients f, k, Γ are as below:

$$\sum_{r=0}^{n}\int_{\eta_r}^{\eta_{r+1}}Y_{r+1}{}^2.dx=1,\qquad f=\sum_{r=0}^{n}\int_{\eta_r}^{\eta_{r+1}}F_{r+1}.Y_{r+1}.dx,\qquad k=1+\sum_{r=1}^{n}\alpha_r.Y_r\big|_{x=\eta_r}{}^2 \tag{24-26}$$

$$\begin{aligned}
\Gamma=&\sum_{r=0}^{n}\int_{\eta_r}^{\eta_{r+1}}\left\langle Y_{r+1}{}'.\phi_{(r+1)1}{}'+2.Y_{r+1}{}'.\phi_{(r+1)2}{}'\right\rangle dx.Y_0{}''+\sum_{r=0}^{n}\int_{\eta_r}^{\eta_{r+1}}\left\langle\frac{3}{2}.Y_{r+1}{}'^{2}+Y_0{}'.\phi_{(r+1)1}{}'+2.Y_0{}'.\phi_{(r+1)2}{}'\right\rangle dx.Y_{m+1}{}''\\
&+\sum_{r=0}^{n}\int_{\eta_r}^{\eta_{r+1}}Y_0{}'.Y_{r+1}{}'dx.\left[\phi_{(m+1)1}{}''+2.\phi_{(m+1)2}{}''\right]
\end{aligned} \tag{27}$$

Let the complex amplitudes A be written as follows:

$$A(T_2)=\frac{1}{2}.a.e^{i.\theta},\qquad \bar{A}(T_2)=\frac{1}{2}.a.e^{-i.\theta},\qquad \theta=\theta(T_2) \tag{28-29}$$

where a is the real amplitude and θ is the phase. Inserting these definitions into Eq. (23), and separating real and imaginary parts, one obtains the following phase-modulation equations:

$$\frac{\mu}{k}.a + \dot{a} = \frac{1}{2.\omega}.\frac{f}{k}.sin\gamma, \qquad -a.\dot{\theta} + \lambda.a^3 = \frac{1}{2.\omega}.\frac{f}{k}.\cos\gamma \tag{30-31}$$

where phase γ and λ, indicating the effects of the nonlinear terms to the natural frequency, can be obtained as below:

$$\lambda = -\frac{1}{8.\omega.k}.\left\{\sum_{r=0}^{n}\int_{\eta_r}^{\eta_{r+1}} Y_{r+1}.\Gamma_{r+1}.dx\right\}, \quad \gamma = \sigma.T_2 - \theta, \tag{32-33}$$

Using Eq. (16), natural frequencies of the curved beam-mass system can be calculated. The first five natural frequencies were given in Tabs. 1 and 2 for different magnitudes and locations of the masses. From these tables, one can see that increasing the magnitudes of the concentrated masses result increase in the natural frequencies. If the number and magnitudes of the masses attached near to the middle of the beam increase, the natural frequencies will increase more.

Free undamped vibration behavior of the system can be examined from the nonlinear frequency-amplitude curves. In order to investigate the steady-state solutions, $\dot{a}$ is assumed to be zero and $f=\mu=\sigma=0$ is taken. Thus, the nonlinear frequency equation defined as below in Eq. (34) can be written as in Eq. (35) under constant amplitude assumption as $a=a_0$;

$$\omega_{nl} = \omega + \dot{\theta}, \quad \omega_{nl} = \omega + \lambda.a_0^{\,2} \tag{34-35}$$

From above equations, relation between the nonlinear frequency and the vibration amplitude is noticed to be of parabolic type.

η_1	η_2	α_1	α_2	ω_1	ω_2	ω_3	ω_4	ω_5	$\lambda(\omega_1)$
0.1	0.3	1	1	7.415	27.830	55.415	99.112	196.791	-0.6176
		1	10	3.016	26.559	51.089	94.530	194.769	-0.2423
		10	1	5.360	19.114	38.637	96.707	195.720	-0.4506
0.1	0.5	1	1	6.741	28.184	54.793	116.644	186.171	-0.5635
		1	10	2.601	27.381	51.553	114.269	182.097	-0.2185
		10	1	5.230	14.057	51.300	108.899	184.774	-0.4731
0.1	0.7	1	1	7.535	22.674	60.319	125.032	174.866	-0.6333
		1	10	3.027	20.376	58.944	124.297	168.195	-0.2449
		10	1	5.630	12.584	51.182	121.441	171.654	-0.6025
0.3	0.5	1	1	5.779	25.166	60.945	141.294	183.112	-0.4812
		1	10	2.538	22.040	54.787	140.869	179.432	-0.2129
		10	1	2.881	18.174	59.622	137.002	180.906	-0.2347
0.3	0.7	1	1	6.343	18.180	83.793	137.955	172.914	-0.5298
		1	10	2.952	13.950	82.956	135.816	167.630	-0.2431
		10	10	2.311	6.373	82.038	134.247	161.940	-0.1933

Table 1: The first five natural frequencies and the effects of the nonlinearity (λ) for the first mode of the curved beam with two concentrated masses.

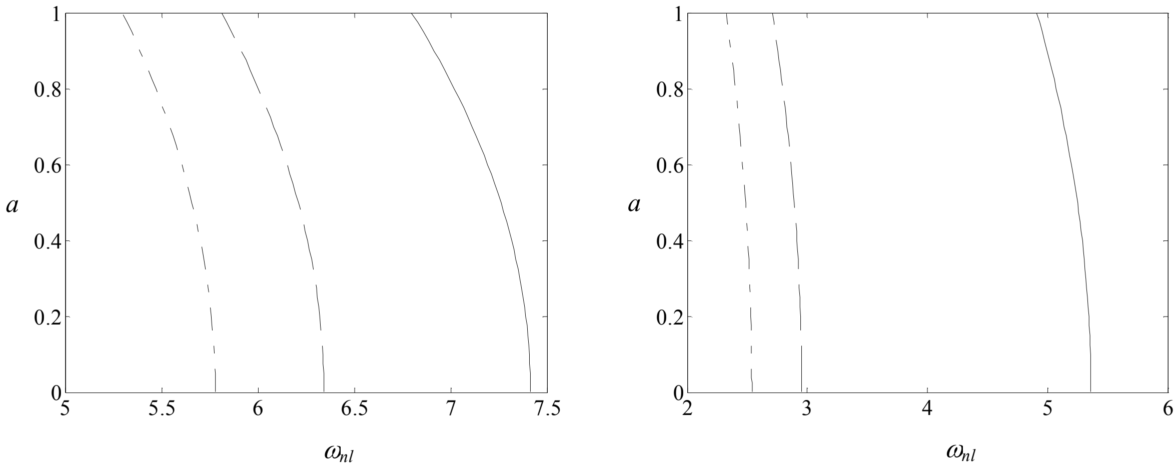
η_1	η_2	η_3	α_1	α_2	α_3	ω_1	ω_2	ω_3	ω_4	ω_5	$\lambda(\omega_1)$
0.1	0.3	0.5	1	1	1	5.645	22.643	52.994	71.324	178.259	-0.4696
			1	1	10	2.528	19.436	51.143	66.114	175.608	-0.2120
			1	10	1	2.862	17.845	50.760	63.359	177.419	-0.2330
			10	10	1	2.700	14.023	24.003	60.998	177.176	-0.2186
			1	10	10	2.039	9.683	50.284	56.843	174.928	-0.1694
			10	1	1	4.647	14.049	38.150	65.455	177.937	-0.4008
0.3	0.5	0.7	1	1	1	5.097	18.180	45.474	137.955	151.362	-0.4250
			1	10	1	2.470	18.180	32.340	137.955	148.660	-0.2071
			1	1	10	2.804	12.675	41.748	135.549	147.998	-0.2320
			10	10	1	2.013	9.243	27.458	135.455	145.672	-0.1675
			10	1	10	2.233	6.373	36.714	134.247	143.636	-0.1866
			10	10	10	1.759	6.373	16.030	134.247	141.485	-0.1467

0.1	0.4	0.8	1	1	1	6.279	18.917	40.669	101.989	193.321	-0.5244
			1	10	1	2.656	17.272	37.368	99.382	189.805	-0.2211
			1	1	10	3.612	11.968	39.452	98.765	193.034	-0.3070
			10	1	1	5.004	13.185	25.970	99.472	186.149	-0.4521
			10	10	1	2.540	12.278	19.465	97.452	181.964	-0.2103
			10	1	10	3.426	8.059	21.966	96.126	185.802	-0.3067

Table 2: The first five natural frequencies and the effects of the nonlinearity (λ) for the first mode of the curved beam with three concentrated masses.

For the first mode of the vibration, λ values indicating the effects of the nonlinear terms to the natural frequency were given in Table 1 and 2. As seen from Eq. (35), for λ values with negative sign nonlinear terms have decreasing effect on the natural frequencies for the first mode. This decreasing effect reduces with increasing both magnitudes and number of the masses.

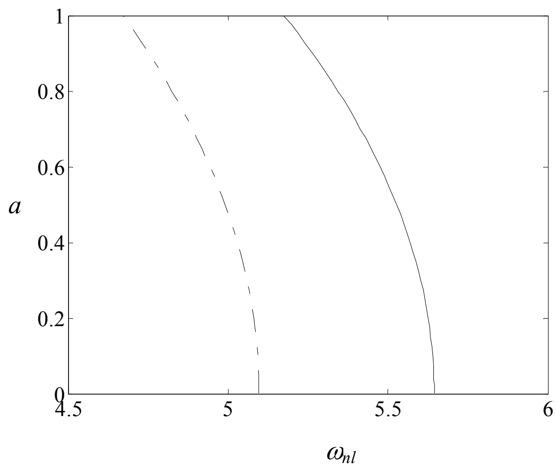
In Figs. (2-3), nonlinear frequency-amplitude curves have been plotted for different number of the masses, mass ratios, and mass locations from the left support. These curves were drawn for the case of two concentrated masses in Fig. 2. In Fig. 2.a, these masses have the same magnitude. Holding the place of one of these masses constant *($\eta_1=0.1$)*, characteristics of the frequency-response curve were investigated by changing the location of the other mass. These masses have different magnitudes in Fig. 2.b. Changing the location of the big mass *($\alpha=10$)*, its effect on the nonlinear frequency was searched. Nonlinear frequency-amplitude curves in Fig. 3 were drawn for the case of three concentrated masses. Having equal masses in magnitude, different mass locations were used for each curve in Fig. 3.a. Thus, the effects of both symmetric and asymmetric cases on the nonlinear frequency-amplitude curves were investigated. In Fig. 3.b, masses in different magnitudes were used and placed on the beam constituting the symmetric and asymmetric cases. As seen from these curves, increasing the number and magnitudes of the concentrated masses decrease both linear and nonlinear frequencies of the system.

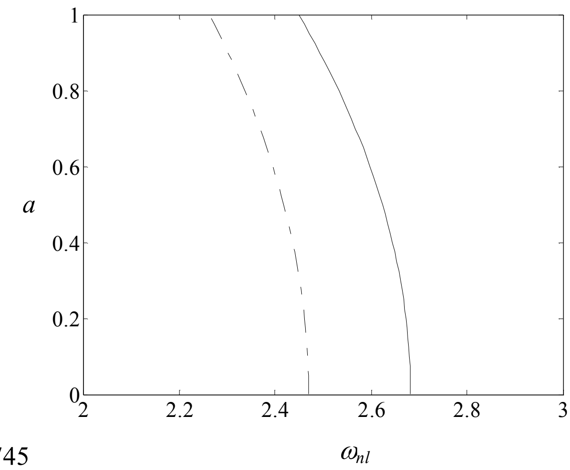


a) —— $\alpha_1=1$, $\alpha_2=1$, $\eta_1=0.1$, $\eta_2=0.3$,
- · - $\alpha_1=1$, $\alpha_2=1$, $\eta_1=0.3$, $\eta_2=0.5$,
- - - $\alpha_1=1$, $\alpha_2=1$, $\eta_1=0.3$, $\eta_2=0.7$

b) —— $\alpha_1=10$, $\alpha_2=1$, $\eta_1=0.1$, $\eta_2=0.3$
- · - $\alpha_1=1$, $\alpha_2=10$, $\eta_1=0.3$, $\eta_2=0.5$
- - - $\alpha_1=1$, $\alpha_2=10$, $\eta_1=0.3$, $\eta_2=0.7$

Figure 2: Nonlinear frequency-amplitude curves for the first mode of the curved beam with two concentrated masses.





a) —— α_1=1, α_2=1, α_3=1, η_1=0.1, η_2=0.3, η_3=0.5
- . - α_1=1, α_2=1, α_3=1, η_1=0.3, η_2=0.5, η_3=0.7

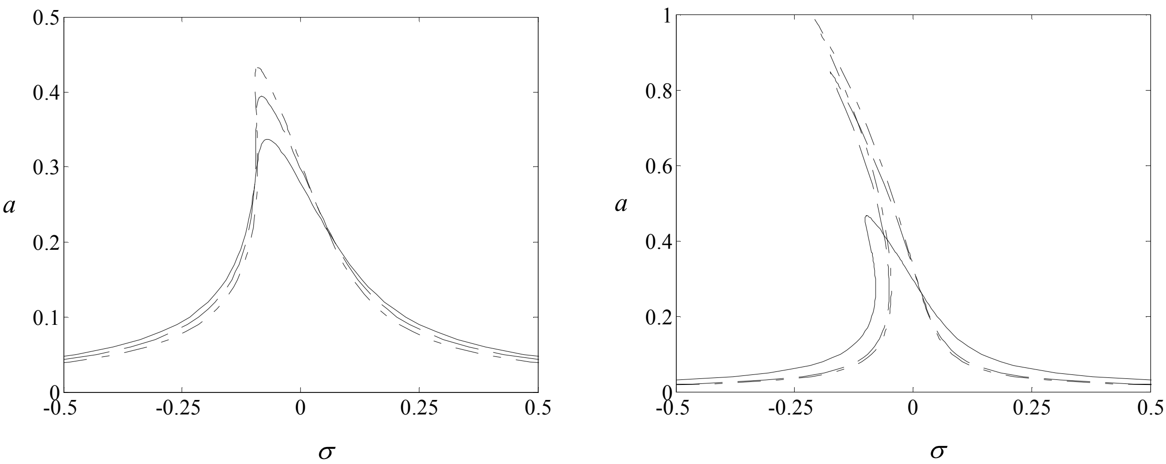
b) —— α_1=1, α_2=10, α_3=1, η_1=0.1, η_2=0.3, η_3=0.5
- . - α_1=1, α_2=10, α_3=1, η_1=0.3, η_2=0.5, η_3=0.7

Figure 3: Nonlinear frequency-amplitude curves for the first mode of the curved beam with three concentrated masses.

For the case of system being damped and externally forced, let us investigate the nonlinear vibration behavior of the system. At steady-state region, $\dot{a}$ and $\dot{\gamma}$ were taken as zero denoting no change in amplitude and phase with time. Eliminating γ in Eqs. (30-31), one can obtain the relation between the detuning parameter (σ) and the amplitude as below:

$$\sigma = \lambda.a_0^{\,2} \pm \sqrt{\left(\frac{f}{2.a_0.\omega.g}\right)^2 - \left(\frac{\mu}{g}\right)^2} \tag{36}$$

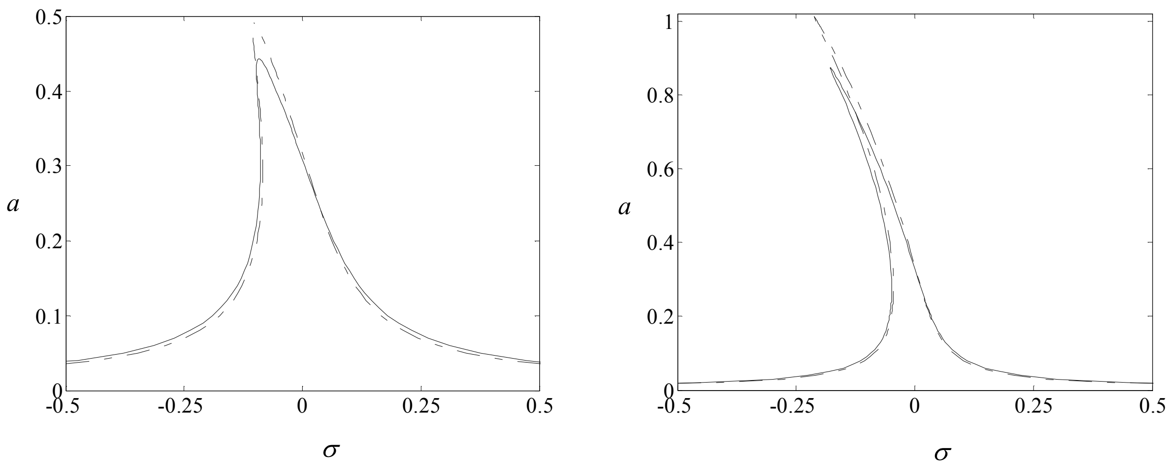
Forced and damped vibrations of the system can be investigated by plotting the frequency-amplitude curves from Eq. (36). These curves were drawn for $f=1$(forcing) and $\mu=0.2$(damping) in Figs. (4-5). Case of two concentrated masses was considered in Fig. 4. In Fig. 4.a, taking the masses equal in magnitude, effects of the mass locations on the frequency-response curves were investigated. In Fig. 4.b, considering different magnitudes of the masses, effects of the big one and its location on the curves were treated. Case of three concentrated masses was considered in Figs. 5. Making up symmetric and asymmetric cases according to different mass locations, their effects on the frequency-response curves were investigated. From these figures, increasing the magnitudes and number of masses the maximum amplitudes of the vibration increase and the system exhibits more softening behavior.



a) —— α_1=1, α_2=1, η_1=0.1, η_2=0.3
- . - α_1=1, α_2=1, η_1=0.3, η_2=0.5
- - - α_1=1, α_2=1, η_1=0.3, η_2=0.7

b) —— α_1=10, α_2=1, η_1=0.1, η_2=0.3
- . - α_1=1, α_2=10, η_1=0.3, η_2=0.5
- - - α_1=1, α_2=10, η_1=0.3, η_2=0.7

Figure 4: Forcing frequency-response curves for the first mode of the curved beam with two concentrated masses.



a) —— $\alpha_1=1,\ \alpha_2=1,\ \alpha_3=1,\ \eta_1=0.1,\ \eta_2=0.3,\ \eta_3=0.5$
– · – $\alpha_1=1,\ \alpha_2=1,\ \alpha_3=1,\ \eta_1=0.3,\ \eta_2=0.5,\ \eta_3=0.7$

b) —— $\alpha_1=1,\ \alpha_2=10,\ \alpha_3=1,\ \eta_1=0.1,\ \eta_2=0.3,\ \eta_3=0.5$
– · – $\alpha_1=1,\ \alpha_2=10,\ \alpha_3=1,\ \eta_1=0.3,\ \eta_2=0.5,\ \eta_3=0.7$

Figure 5: Forcing frequency-response curves for the first mode of the curved beam with three concentrated masses.

Results

In this study, nonlinear vibrations of a curved beam carrying multiple concentrated masses were investigated. Beam was assumed Euler-Bernoulli type and sinusoidal function was used for the curvature of the beam. Primary resonance case was investigated. Approximate solutions were obtained by means of the method of multiple scales, a perturbation technique. In perturbation series, the first order corresponds to the linear problem of the system. Including effects of the nonlinear terms to the linear solution at other orders, the nonlinear system was solved. For the steady-state case, free-undamped and forced-damped vibrations were investigated. Effects of the magnitudes, locations and number of concentrated masses on nonlinear vibrations were analyzed in detail.

In the primary resonance case, nonlinear effects result in softening behavior of the curved beam-mass system. Such a behavior enables nonlinear frequencies to decrease with amplitude in free-undamped vibrations, and those frequency-response curves to bend to the left in the forced-damped vibrations. Softening behavior was observed to increase with increasing mass ratios and mass numbers. In this case, the nonlinear frequencies decrease, the region of jump phenomena expands, and the maximum amplitudes increase. Same behavior was seen in the case of masses being placed to the middle point of the beam instead of ends of the beam.

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